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Some simple properties of hyperspherical solid harmonics

Dr.ir. D.J. Hofsommer

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door

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1. Consider Laplace's equation in four dimensions

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} + \frac{\partial^2 v}{\partial x_4^2} = 0 \qquad 1.1)$$

and substitute $x_1 = \sqrt{y}\cos\varphi$

$x_2 = \sqrt{y}\sin\varphi$

$x_3 = \sqrt{z}\cos\psi$

$x_4 = \sqrt{z}\sin\psi$

the transformed equation is

$$\frac{\partial}{\partial y}\left(y\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial v}{\partial z}\right) + \frac{1}{4y}\frac{\partial^2 v}{\partial\varphi^2} + \frac{1}{4z}\frac{\partial^2 v}{\partial\psi^2} = 0$$

Separate according to $V(y,z,\varphi,\psi) = U(y,z)\,\phi(\varphi)\Psi(\psi)$. We get the equations

$$\frac{d^2\phi}{d\varphi^2} + 4m^2\phi = 0 \qquad \frac{d^2\Psi}{d\psi^2} + 4n^2\Psi = 0$$

$$\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial U}{\partial z}\right) - \left(\frac{m^2}{y} + \frac{n^2}{z}\right)U = 0 \qquad 1.2)$$

Any homogeneous solution of (1.2) i.e. any solution satisfying

$$y\frac{\partial U}{\partial y} + z\frac{\partial U}{\partial z} = kU \qquad 1.3)$$

will be called a hyperspherical harmonic:

We will restrict ourselves to those values of k,m,n for which the four quantities $k \pm m \pm n$ are nonnegative integers. From (1.2) and (1.3) either the derivatives with respect to y or to z may be eliminated. The result is an ordinary second order differential equation. This shows that the pair of equations (2) and (3) has two linear independent solutions.

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In order to solve (1.2) and (1.3) we substitute

$$U = y^{-m} z^{-n} W$$

Then $W(y,z)$ satisfies

$$\frac{\partial}{\partial y}\left(y\frac{\partial W}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial W}{\partial z}\right) - 2m\frac{\partial W}{\partial y} - 2n\frac{\partial W}{\partial z} = 0 \qquad 1.4)$$

$$y\frac{\partial W}{\partial y} + z\frac{\partial W}{\partial z} = (k + m + n)W \qquad 1.5)$$

We now try to find a function $W(y,z)$ satisfying the initial conditions $W(0,0) = 0$. Applying a Laplace transformation we find, if $W(y,z) \doteqdot \overline{w}(p,q)$, that

$$p^2 \frac{\partial \overline{w}}{\partial p} + q^2 \frac{\partial \overline{w}}{\partial q} + (2mp + 2nq)\overline{w} = 0 \qquad 1.6)$$

which equation is solved by

$$\overline{w} = \frac{p^{-2m} q^{-2n}}{(k - m - n)!} f\left(\frac{1}{p} - \frac{1}{q}\right) \qquad 1.7)$$

and $$p\frac{\partial \overline{w}}{\partial p} + q\frac{\partial \overline{w}}{\partial q} = -(k + m + n)\overline{w} \qquad 1.8)$$

which equation specializes solution (1.7) into

$$\overline{w} = \frac{p^{-2m} q^{-2n}}{(k - m - n)!} \left(\frac{1}{p} - \frac{1}{q}\right)^{k - m - n}$$

$$= \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(k-m-n-s)!} \left(\frac{1}{p}\right)^{k+m-n-s} \left(\frac{1}{q}\right)^{2n+s} \qquad 1.9)$$

Hence $$W(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(k-m-n-s)!(k+m-n-s)!(2n+s)!} y^{k+m-n-s} z^{2n+s}$$

The corresponding $U(y,z)$ will be denoted by $P_k^{-m,-n}(y,z)$

$$P_k^{-m,-n}(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(s+2n)!(k+m-n-s)!(k-m-n-s)!} y^{k-n-s} z^{s+n}$$

or $$P_k^{m,n}(y,z) = \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(s-2n)!(k+m+n-s)!(k-m+n-s)!} y^{k+n-s} z^{s-n} \qquad 1.10)$$

Because of the factors $(s-2n)!(k-m+n-s)!$ in the denominator the terms for which $0 \leq s < 2n$ and $k-m+n < s \leq k+m+n$ vanish. Replacing s by $s+2n$ we find

$$P_k^{m,n}(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^{s+2n}}{(s+2n)!s!(k+m-n-s)!(k-m-n-s)!} y^{k-n-s} z^{s+n}$$

$$= (-1)^{2n} P_k^{-m,-n}(y,z) \qquad 1.11)$$

In the same way we find

$$P_k^{m,n}(y,z) = P_k^{-m,n}(y,z) = (-1)^{2n} P_k^{m,-n}(y,z) = (-1)^{2n} P_k^{-m,-n}(y,z) \qquad 1.12)$$

If in (1.10) we replace s by k+m+n-s it is easily seen that

$$P_k^{m,n}(y,z) = (-1)^{k+m+n} P_k^{n,m}(z,y) \qquad 1.13)$$

Formula (1.10) can be easily generalized to a form valid for arbitrary values of k,m,n. This generalization is

$$P_k^{m,n}(y,z) = \frac{y^k}{\prod_n (-2n) \prod (k+m+n) \prod (k-m+n)} \cdot$$
$$\cdot \left(\frac{y}{z}\right) F\left(-k-m-n, -k+m-n; 1-2n; -\frac{z}{y}\right) \qquad 1.14)$$

We have found

$$P_k^{m,n}(y,z) = \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(s-2n)!(k+m+n-s)!(k-m+n-s)!} y^{k+n-s} z^{s-n} \qquad 1.10)$$

Hence

$$y^{-m} z^{-n} P_k^{m,n}(y,z) = \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(k+m+n-s)!} \left(\frac{1}{p}\right)^{k-m+n-s} \left(\frac{1}{q}\right)^{s-2n} \qquad 1.15)$$

$$= \frac{p^{2m} q^{2n}}{(k+m+n)!} \left(\frac{1}{p} - \frac{1}{q}\right)^{k+m+n} \qquad 1.16)$$

$$= \frac{p^{2m} q^{2n}}{(k+m+n)!} (q-p)^{k+m+n} (pq)^{-k-m-n} \qquad 1.17)$$

and therefore

$$P_k^{m,n}(y,z) = y^m z^n \frac{\partial^{2(m+n)}}{\partial y^m \partial z^n} P_{k+m+n}^{0,0}(y,z) \qquad 1.18)$$

since, from (1.16)

$$P_{k+m+n}^{0,0}(y,z) = \frac{1}{(k+m+n)!} \left(\frac{1}{p} - \frac{1}{q}\right)^{k+m+n}$$

Again, from (1.17)

$$P_k^{m,n}(y,z) = \frac{1}{(k+m+n)!^3} y^m z^n \frac{\partial^{2(m+n)}}{\partial y^m \partial z^n} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right)^{k+m+n} (yz)^{k+m+n} \qquad 1.19)$$

We also have

$$y^m z^n P_k^{m,n}(y,z) = (-1)^{2n} y^m z^n P_k^{-m,-n}(y,z)$$
$$= \frac{(-1)^{2n} p^{-2m} q^{-2n}}{(k-m-n)!} (q-p)^{k-m-n} (pq)^{-k+m+n}$$

$$\doteqdot \frac{(-1)^{2n}p^{2n}q^{2m}}{(k-m-n)!}(q-p)^{k-m-n}(pq)^{-k-m-n}$$

or $P_k^{m,n}(y,z) =$

$$\frac{(-1)^{2n}y^{-m}z^{-n}}{(k+m+n)!^2(k-m-n)!}\frac{\partial^{2(m+n)}}{\partial y^n \partial z^m}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial y}\right)^{k-m-n}(yz)^{k+m+n} \qquad 1.20)$$

Finally, if m = n we find from (1.17)

$$y^{-m}z^{-m}P_k^{m,n}(y,z) \doteqdot \frac{1}{(k+2m)!}(q-p)^{k+2m}(pq)^{-k}$$

or $$P_k^{m,m}(y,z) = \frac{(yz)^m}{k!^2(k+2m)!}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial y}\right)^{k+2m}(yz)^k \qquad 1.21)$$

Having established the image of $y^m z^n P_k^{m,n}(y,z)$ it is easy to find numerous recurrent relations. We derive a few.

From

$$(k+m+n)\frac{p^{2m}q^{2n}}{(k+m+n)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n} =$$

$$\frac{p^{2m-1}q^{2n}}{(k+m+n-1)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n-1} - \frac{p^{2m}q^{2n-1}}{(k+m+n-1)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n-1}$$

follows

$$(k+m+n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}(y.z) - \sqrt{z}P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}(y,z) \qquad 1.22)$$

and by aid of (1.12)

$$(k-m+n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m+\frac{1}{2},n}(y,z) - \sqrt{z}P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}(y,z) \qquad 1.23)$$

$$(k+m-n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}(y,z) + \sqrt{z}P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}}(y,z) \qquad 1.24)$$

$$(k-m-n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m+\frac{1}{2},n}(y,z) + \sqrt{z}P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}}(y,z) \qquad 1.25)$$

Addition of (1.22) and (1.25) yields

$$2kP_k^{m,n}(y,z) = \sqrt{y}\left(P_{k-\frac{1}{2}}^{m+\frac{1}{2},n} + P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}\right) + \sqrt{z}\left(P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}} - P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}\right) \qquad 1.26)$$

2. The Laplce equation (1.1) has a solution

$$V = f(x_1 - ix_4)$$

Let

$$V = 2^{2k}(x_1 - ix_4)^{2k} = (\sqrt{y}e^{i\varphi} + \sqrt{y}e^{-i\varphi} - \sqrt{z}e^{i\psi} + \sqrt{z}e^{-i\psi})^{2k} \qquad 2.1)$$

Then

$$V = (2k)! \sum_{m,n} P_k^{m,n}(y,z)e^{2i(m\varphi + n\psi)} \qquad 2.2)$$

Assume this formula to be valid for k. Then

$$2^{2k+1}(x_1 - ix_4)^{2k+1} =$$

$$(2k)!(\sqrt{y}e^{i\varphi} + \sqrt{y}e^{-i\varphi} - \sqrt{z}e^{i\psi} + \sqrt{z}e^{-i\psi}) \sum P_k^{m,n}(y,z)e^{2i(m\varphi + n\psi)}$$

$$= (2k)! \sum P_k^{m,n}\Big[\sqrt{y}\Big\{e^{2i\{(m+\frac{1}{2})\varphi + n\psi\}} + e^{-2i\{(m-\frac{1}{2})+n\psi\}}\Big\}$$

$$- \sqrt{z}\Big\{e^{2i\{m\varphi + (n+\frac{1}{2})\psi\}} - e^{-2i\{m\varphi + (n-\frac{1}{2})\psi\}}\Big\}\Big]$$

$$= (2k)! \sum \Big[\sqrt{y}(P_k^{m-\frac{1}{2},n} + P_k^{m+\frac{1}{2},n}) - \sqrt{z}(P_k^{m,n-\frac{1}{2}} - P_k^{m,n+\frac{1}{2}})\Big] e^{2i(m\varphi + n\psi)}$$

$$= (2k+1)! \sum P_{k+\frac{1}{2}}^{m,n}(y,z)e^{2i(m\varphi + n\psi)}$$

because of (1.26)

It is easily seen from (1.10) that $P_0^{0,0} = 1$. Hence our formula is valid for k = 0 and therefore for all values of k

3. Another approach is to replace (1.2) by a set of aequivalent differo-difference equations. Consider the set

$$\frac{\partial}{\partial y} y^m U_k^{m,n} = y^{m-\frac{1}{2}} U_{k-\frac{1}{2}}^{m-\frac{1}{2},n} \qquad \frac{\partial}{\partial z} z^n U_k^{m,n} = -z^{n-\frac{1}{2}} U_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}$$

$$\frac{\partial}{\partial y} y^{-m} U_k^{m,n} = y^{-m-\frac{1}{2}} U_{k-\frac{1}{2}}^{m+\frac{1}{2},n} \qquad \frac{\partial}{\partial z} z^{-n} U_k^{m,n} = z^{-n-\frac{1}{2}} U_{k-\frac{1}{2}}^{m,n+\frac{1}{2}} \tag{3.1}$$

Indeed we have

$$y^m \frac{\partial}{\partial y} \left(y^{1-2m} \frac{\partial}{\partial y} y^m U_k^{m,n}\right) = y^m \frac{\partial}{\partial y} \left(y^{1-m} \frac{\partial}{\partial y} U_k^{m,n} + m y^{-m} U_k^{m,n}\right)$$

$$= y \frac{\partial^2}{\partial y^2} U_k^{m,n} + \frac{\partial}{\partial y} U_k^{m,n} - \frac{m^2}{y} U_k^{m,n}$$

but also

$$y^m \frac{\partial}{\partial y} \left(y^{1-2m} \frac{\partial}{\partial y} y^m U_k^{m,n}\right) = y^m \frac{\partial}{\partial y} y^{-m+\frac{1}{2}} U_{k-\frac{1}{2}}^{m-\frac{1}{2},n} = U_{k-1}^{m,n}$$

In the same way we find

$$z^n \frac{\partial}{\partial z}\left(z^{1-2n} \frac{\partial}{\partial z} z^n U_k^{m,n}\right) = z \frac{\partial^2}{\partial z^2} U_k^{m,n} + \frac{\partial}{\partial z} U_k^{m,n} - \frac{n^2}{z} U_k^{m,n} = -U_{k-1}^{m,n}$$

which shows, that any function which satisfies the set (3.1) is a solution of equation (1.2)

Reading the lower formulae (3.1) the other way round we find

$$\begin{aligned} y^{-m} z^{-n} U_k^{m,n} &= z^{-n} \frac{\partial}{\partial y} y^{-m+\frac{1}{2}} U_{k+\frac{1}{2}}^{m-\frac{1}{2},n} \\ &= z^{-n} \frac{\partial^{2m}}{\partial y^{2m}} U_{k+m}^{0,n} \\ &= \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial}{\partial z} z^{-n+\frac{1}{2}} U_{k+m+\frac{1}{2}}^{0,n-\frac{1}{2}} \\ &= \frac{\partial^{2m+2n}}{\partial y^{2m} \partial z^{2n}} U_{k+m+n}^{0,0} \end{aligned} \tag{3.2}$$

Hence every solution of (1.2) can be derived by differentiating the solutions of

$$\frac{\partial}{\partial y}\left(y \frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(z \frac{\partial U}{\partial z}\right) = 0 \tag{3.3}$$

The homogeneous solutions of degree k+m+n of this equation are

$$P_k^{0,0}(y,z) = \sum_{s=0}^{k} \frac{1}{s!^2 (k-s)!^2} y^{k-s} (-z)^s$$

$$Q_k^{0,0}(y,z) = \sum_{s=0}^{k} \frac{\ln z - \ln y + 2\psi(k-s) - 2\psi(s)}{s!^2 (k-s)!^2} y^{k-s} (-z)^s \tag{3.4}$$

$$= \ln \frac{z}{y} P_k^{0,0} + 2 \sum_{s=0}^{k} \frac{\psi(k-s) - \psi(s)}{s!^2 (k-s)!^2} y^{k-s} (-z)^s$$

The functions $P_k^{m,n}(y,z)$ follow from (3.4) by means of the differentiating process (3.2)

A special value is

$$Q_0^{0,0}(y,z) = \ln\frac{z}{y} \qquad 3.5)$$

4. Finally we establish some relations between hyperspherical harmonics and Bessel's functions. It is seen without difficulty that (1.2) is satisfied by a product of Bessel's ordinary and modified functions

The image of Bessel's functions is given by

$$y^m J_{2m}(2\sqrt{y}) \doteqdot p^{-2m} e^{-\frac{1}{p}} \qquad y^m I_{2m}(2\sqrt{y}) \doteqdot p^{-2m} e^{\frac{1}{p}}$$

Hence

$$\begin{aligned} y^m z^n I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) &\doteqdot p^{-2m} q^{-2n} e^{\frac{1}{p} - \frac{1}{q}} \\ &\doteqdot \sum_{s=0}^{\infty} \frac{p^{-2m} q^{-2n}}{s!} \left(\frac{1}{p} - \frac{1}{q}\right)^s \\ &= y^m z^n \sum_{s=0}^{\infty} (-1)^{2n} P^{m,n}_{s+m+n}(y,z) \end{aligned}$$

or $$I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = (-1)^{2n} \sum_{s=0}^{\infty} P^{m,n}_{s+m+n}(y,z) \qquad 4.1)$$

In the same way

$$\begin{aligned} y^m z^n J_{2m}(2\sqrt{y}) I_{2n}(2\sqrt{z}) &\doteqdot p^{-2m} q^{-2n} e^{-\frac{1}{p} + \frac{1}{q}} \\ &\doteqdot \sum_{s=0}^{\infty} \frac{p^{-2m} q^{-2n}}{s!} (-1)^s \left(\frac{1}{p} - \frac{1}{q}\right)^s \\ &= y^m z^n (-1)^{2n} \sum_{s=0}^{\infty} (-1)^s P^{m,n}_{s+m+n}(y,z) \end{aligned}$$

or $$J_{2m}(2\sqrt{y}) I_{2n}(2\sqrt{z}) = (-1)^{2n} \sum_{s=0}^{\infty} (-1)^s P^{m,n}_{s+m+n}(y,z) \qquad 4.2)$$

According to (4.1) we have

$$I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n}$$

$$\frac{\partial}{\partial m} I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t [\ln y - 2\psi(s+2m-t)]}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n} \qquad 4.3)$$

$$\frac{\partial}{\partial n} I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t [\ln z - 2\psi(t+2n)]}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n} \qquad 4.4)$$

In (4.3) and (4.4) put m=n=0 and subtract (4.3) from (4.4). Then by use of the known relations

$$\left[\frac{\partial}{\partial n} J_n(z)\right]_{n=0} = \tfrac{1}{2} N_0(z), \quad \left[\frac{\partial}{\partial n} I_n(z)\right]_{n=0} = \tfrac{1}{2} K_0(z)$$

we find

$$I_0(2\sqrt{y})N_0(2\sqrt{z}) - K_0(2\sqrt{y})J_0(2\sqrt{z}) =$$

$$\sum_{s=0}^{\infty}\sum_{t=0}^{s}(-1)^t\,\frac{\ln z-\ln y+2\psi(s-t)-2\psi(t)}{t!^2(s-t)!^2}\;y^{s-t}z^t$$

$$=\sum_{s=0}^{\infty}Q_s^{0,0}(y,z)$$

Apply the operator $\dfrac{\partial^{2m+2n}}{\partial y^{2m}\partial z^{2n}}$ to both sides of this equation. Then by virtue of

$$\frac{\partial^{2n}}{\partial y^{2n}}J_0(2\sqrt{y}) = (-1)^{2n}y^{-n}J_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}}N_0(2\sqrt{y}) = (-1)^{2n}y^{-n}N_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}}I_0(2\sqrt{y}) = y^{-n}I_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}}K_0(2\sqrt{y}) = y^{-n}K_{2n}(2\sqrt{y})$$

We have

$$(-1)^{2n}y^{-m}z^{-n}\left[I_{2m}(2\sqrt{y})N_{2n}(2\sqrt{z}) - K_{2m}(2\sqrt{y})J_{2n}(2\sqrt{z})\right] =$$

$$= y^{-m}z^{-n}\sum_{s=0}^{\infty}Q_{s-m-n}^{m,n}(y,z)$$

or

$$I_{2m}(2\sqrt{y})N_{2n}(2\sqrt{z}) - K_{2m}(2\sqrt{y})J_{2n}(2\sqrt{z}) =$$

$$(-1)^{2n}\sum_{s=0}^{\infty}Q_{s+m+n}^{m,n}(y,z) \qquad 4.5)$$

because

$$Q_k^{m,n}(y,z) = 0 \text{ if } k < m+n$$

In the same way we derive

$$J_{2m}(2\sqrt{y})K_{2n}(2\sqrt{z}) - N_{2m}(2\sqrt{y})I_0(2\sqrt{z}) =$$

$$(-1)^{2n}\sum_{s=0}^{\infty}(-1)^s\,Q_{s+m+n}^{m,n}(y,z) \qquad 4.6$$